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Exact solutions to the ideal magneto-gas-dynamics equations through Lie group analysis and substitution principles

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Abstract

In this paper, we consider the equations governing an inviscid, thermally non-conducting fluid of infinite electrical conductivity in the presence of a magnetic field and subject to no extraneous force. By using conveniently the Lie point symmetries admitted, we map the governing system into an equivalent autonomous form. The transformed system can be directly inspected in order to find some simple solutions that, written in terms of the original variables, provide non-trivial exact solutions of the system at hand. Use is made of some finite transformations known in the literature as substitution principles, enabling us to build exact solutions containing some arbitrary functions. The link between the substitution principle and the recently discovered Bogoyavlenskij symmetries for equilibrium magnetohydrodynamics is also discussed. Some of the recovered solutions are considered to solve well-known physically relevant boundary value problems and the linear stability analysis is performed, thus generalizing well-established results.

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1. Introduction

The explicit determination of exact solutions to systems of partial differential equations (PDEs) of physical interest is an important task, especially when the solutions contain arbitrary functions useful for solving initial and/or boundary value problems. One of the most powerful methods in order to determine particular solutions to PDEs is based upon the study of their invariance with respect to one-parameter Lie group of point transformations [1–7] allowing for the determination of similarity solutions: this method is also referred to as Lie direct method.

In contrast, one may address the inverse problem, which consists in finding the constraints to be imposed to a given set of PDEs in order to have their invariance with respect to an assigned family of transformations; for instance, this is the case of the results known in literature as substitution principles, originally introduced for the steady equations of ideal gas dynamics [8, 9] and of the steady magneto-gas-dynamics [10] when a separable equation of state is taken or of the symmetries recently found by Bogoyavlenskij [11] for the equilibrium equations of magnetohydrodynamics. Substitution principles have also been given for the unsteady equations of ideal gas dynamics having the pressure steady [12] and for the unsteady equations of ideal magneto-gas-dynamics having the total magnetic pressure steady [13].

The substitution principle for equations of ideal gas dynamics has also been revisited [14, 15] within the context of Lie groups of infinitesimal transformations. As a consequence, some generalizations of the substitution principle given by Smith [12] have been provided for the unsteady equations of ideal gas dynamics and a new substitution principle [15] was stated for the n-dimensional flow of a perfect gas with the adiabatic index equal to $\frac{n+2}{n}$. Furthermore, various classes of exact solutions have been determined [16–18].

The substitution principles for the steady as well as unsteady equations of magneto-gas-dynamics have been considered within the framework of Lie group analysis in [19], where some generalizations have been provided: in particular, a new substitution principle for a plane flow of a fluid with adiabatic index equal to 2 and subjected to a transverse magnetic field has been established. The transformations for the steady equations of magnetohydrodynamics introduced by Bogoyavlenskij [11] contain as a special subcase the substitution principle for steady incompressible magneto-gas-dynamics; also, Cheviakov [20] proved that these Bogoyavlenskij symmetries can be derived within the context of Lie group analysis; up to now, unfortunately, no extension of Bogoyavlevskij symmetries to the unsteady case is known.

In the present paper, we determine some classes of exact solutions to the equations of ideal magneto-gas-dynamics by using in a suitable way the admitted Lie point symmetries [7]. Also, for some of these solutions either the substitution principles or the Bogoyavlevskij transformations can be applied, and this enables us to obtain solutions that involve some arbitrary functions; this may provide some useful degrees of freedom in solving initial and/or boundary value problems of physical interest.

More precisely, by using the Lie point symmetries of the governing equations, we construct some invertible transformations suitable to map the equations at hand to a new equivalent autonomous form [21, 22], from which we may obtain by inspection some exact solutions that, written in terms of the original variables, may represent non-trivial physically meaningful solutions; then the substitution principles or the Bogoyavlenskij transformations can be used with some of these solutions and new solutions are constructed.

In section 2, we report the Lie symmetries admitted by the equations of ideal magneto-gas-dynamics and briefly sketch the statements of the theorems concerned with the substitution principles and the Bogoyavlenskij symmetries that will be used through the rest of the paper. In sections 3 and 4, we consider steady and unsteady ideal magneto-gas-dynamics equations, respectively, and derive some exact explicit solutions depending upon two or three space variables. In section 5, we look for solutions describing a plane motion of a fluid with the adiabatic index equal to 2 and subjected to a transverse magnetic field. In section 6, we consider some boundary value problems of physical interest and investigate the linear stability of some solutions we obtained; some results quoted in [23] are generalized. Finally, section 7 presents some concluding remarks.

2. Invariance of the equations

The equations governing the 3D flow of an inviscid, thermally non-conducting fluid of infinite electrical conductivity in the presence of a magnetic field and subject to no extraneous force are

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \qquad \frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s = 0, \\
&\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) + \nabla p + \mu \mathbf{H} \times (\nabla \times \mathbf{H}) = 0, \\
&\frac{\partial \mathbf{H}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{H}) = 0, \qquad \nabla \cdot \mathbf{H} = 0,
\end{aligned}
\tag{1}
$$

where $\rho(t, \mathbf{x})$ is the mass density, $p(t, \mathbf{x})$ is the pressure, $s(t, \mathbf{x})$ (a function of) is the entropy, μ is the constant magnetic permeability, $\mathbf{v}(t, \mathbf{x}) \equiv (v_1, v_2, v_3)$ and $\mathbf{H}(t, \mathbf{x}) \equiv (H_1, H_2, H_3)$ are the velocity and magnetic fields, respectively, $\mathbf{x} \equiv (x_1, x_2, x_3)$ is the rectangular Cartesian coordinates and t is the time. In the following we shall also consider the 2D version of equations (1) (namely, the equations where the field variables are assumed independent of x_3 and where $v_3 = H_3 = 0$).

The system (1) is supplemented by the equation of perfect gases

$$
\rho = p^{1/\Gamma} \cdot s, \tag{2}
$$

where Γ is the adiabatic index.

The Lie groups of point transformations that leave the system (1) invariant [7] span a $(n^2 + 3n + 8)/2$-dimensional Lie algebra, where n (that will assume in the following the values 2 or 3) is the number of space variables, generated by the following vector fields:

$$
\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial t}, \qquad \Xi_{k+1} = \frac{\partial}{\partial x_k}, \qquad \Xi_{n+2} = t\frac{\partial}{\partial t} + \sum_{k=1}^{n} x_k \frac{\partial}{\partial x_k}, \\
&\Xi_{3+n} = -t\frac{\partial}{\partial t} + \sum_{k=1}^{n} v_k \frac{\partial}{\partial v_k} + \frac{2\Gamma}{\Gamma - 1}\left(p\frac{\partial}{\partial p} + \sum_{k=1}^{n} \frac{H_k}{2} \frac{\partial}{\partial H_k} \right), \\
&\Xi_{4+n} = \frac{\Gamma}{\Gamma - 1}\left(p\frac{\partial}{\partial p} + \sum_{k=1}^{n} \frac{H_k}{2} \frac{\partial}{\partial H_k} \right) + s\frac{\partial}{\partial s}, \\
&\Xi_{4+n+k} = t\frac{\partial}{\partial x_k} + \frac{\partial}{\partial v_k}, \\
&\Xi_{2n+4+l} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} + v_i \frac{\partial}{\partial v_j} - v_j \frac{\partial}{\partial v_i} + H_i \frac{\partial}{\partial H_j} - H_j \frac{\partial}{\partial H_i},
\end{aligned}
\tag{3}
$$

with $i, j, k = 1, \ldots, n$, $j > i$, $l = 1, \ldots, n(n-1)/2$. The operators $\Xi_1, \ldots, \Xi_{n+1}$ characterize time and space translations, Ξ_{n+2}, Ξ_{n+3} and Ξ_{n+4} stretching transformations, $\Xi_{n+5}, \ldots, \Xi_{2n+4}$ the Galilean transformations; finally, the remaining operators characterize the spatial rotations.

It is worth noting the case where we consider a plane flow with a transverse magnetic field, i.e.,

$$
\mathbf{v} = v_1(x_1, x_2, t)\, \mathbf{e}_1 + v_2(x_1, x_2, t)\, \mathbf{e}_2, \qquad \mathbf{H} = h(x_1, x_2, t)\, \mathbf{e}_3, \tag{4}
$$

where $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are the unit vectors along the axes x_1, x_2 and x_3, respectively, and $\Gamma = 2$. The operators of the admitted Lie point symmetries (see [24]) are

$$
\begin{aligned}
&\Xi_1 = \frac{\partial}{\partial t}, \qquad \Xi_2 = \frac{\partial}{\partial x_1}, \qquad \Xi_3 = \frac{\partial}{\partial x_2},\\
&\Xi_4 = t\frac{\partial}{\partial t} + x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2},\\
&\Xi_5 = -t\frac{\partial}{\partial t} + v_1\frac{\partial}{\partial v_1} + v_2\frac{\partial}{\partial v_2} + 4\left(p\frac{\partial}{\partial p} + \frac{h}{2}\frac{\partial}{\partial h}\right),\\
&\Xi_6 = 2\left(p\frac{\partial}{\partial p} + \frac{h}{2}\frac{\partial}{\partial h}\right) + s\frac{\partial}{\partial s},\\
&\Xi_7 = t\frac{\partial}{\partial x_1} + \frac{\partial}{\partial v_1}, \qquad \Xi_8 = t\frac{\partial}{\partial x_2} + \frac{\partial}{\partial v_2},\\
&\Xi_9 = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + v_2\frac{\partial}{\partial v_1} - v_1\frac{\partial}{\partial v_2},\\
&\Xi_{10} = t^2\frac{\partial}{\partial t} + \sum_{i=1}^{2}\left(tx_i\frac{\partial}{\partial x_i} + (x_i - v_i t)\frac{\partial}{\partial v_i}\right) - 4t\left(p\frac{\partial}{\partial p} + \frac{h}{2}\frac{\partial}{\partial h}\right),\\
&\Xi_{11} = F\left(\frac{p}{h^2}, s\right)\left(2p\frac{\partial}{\partial p} - s\frac{\partial}{\partial s} - \frac{2p}{\mu h}\frac{\partial}{\partial h}\right),
\end{aligned}
$$

where $F\left(\frac{p}{h^2}, s\right)$ is an arbitrary function of the indicated arguments.

The infinitesimal operator Ξ_{10}, characterizing the projective group, will prove very useful in the subsequent analysis; in fact, by considering the similarity solutions left invariant by the projective transformation, the reduced system has the same form as the steady system (i.e., the system in which the field variables do not depend on t). This means that if we know a steady solution describing a plane flow with a transverse magnetic field and $\Gamma = 2$, then we get immediately an unsteady solution.

In view of the analysis that will be pursued in the following, let us also consider the Lie groups of point transformations that leave the steady equations of perfect gases invariant [7]; the admitted vector fields

$$
\begin{aligned}
&\Xi_k = \frac{\partial}{\partial x_k}, \qquad \Xi_{n+1} = \sum_{k=1}^{n} x_k\frac{\partial}{\partial x_k},\\
&\Xi_{n+2} = \sum_{k=1}^{n} v_k\frac{\partial}{\partial v_k} + \frac{2\Gamma}{\Gamma - 1}\left(p\frac{\partial}{\partial p} + \sum_{k=1}^{n}\frac{H_k}{2}\frac{\partial}{\partial H_k}\right),\\
&\Xi_{n+3} = \sum_{k=1}^{n} v_k\frac{\partial}{\partial v_k} - 2s\frac{\partial}{\partial s},\\
&\Xi_{n+3+l} = x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i} + v_i\frac{\partial}{\partial v_j} - v_j\frac{\partial}{\partial v_i} + H_i\frac{\partial}{\partial H_j} - H_j\frac{\partial}{\partial H_i},
\end{aligned}
\tag{5}
$$

with $i, j, k = 1, \ldots, n$, $j > i$, $l = 1, \ldots, n(n-1)/2$, generate a $(n^2 + n + 6)/2$-dimensional Lie algebra.

Usually, the knowledge of the Lie point symmetries admitted by a system of PDEs is employed to characterize classes of invariant solutions. But, one may use them to introduce suitable invertible point transformations allowing one to map a given source system of PDEs to an equivalent form (target system) for which classes of exact solutions may be found by simple inspection. The latter task may be accomplished, for instance, by using a result proved in [22] (see also [21]): what is needed is the invariance of the equations at hand with respect to q independent Lie groups of point transformations whose infinitesimal operators generate an Abelian Lie algebra, where q is equal to the number of independent variables.

The invertible point transformation, which is built by considering the canonical variables associated with the q infinitesimal operators, maps the system to an equivalent autonomous form if the source system is not autonomous. Remarkably, the theorem can also be applied when the source system is autonomous: in this case, we get an equivalent autonomous system. The transformation of an autonomous system to an equivalent autonomous one can give, in a simple and systematic way, non-trivial solutions: for instance, the constant solutions of the original autonomous system may be trivial, conversely the constant solutions of the transformed autonomous system may give non-trivial non-constant solutions of the original system (see for instance [25] where this procedure is used to study the propagation of discontinuity waves in non-constant states described by self-similar solution as a propagation problem in a constant state of a transformed system). The procedure to be applied requires a lot of lengthy though straightforward calculations, but it can be rendered quite automatic by using computer algebra packages oriented towards Lie symmetries of differential equations [26–28]: all that is required is to determine the canonical variables of a suitable subset of the Lie groups of invariance and rewrite the differential equations in terms of different sets of variables.

As remarked above, in the applications it is important to determine solutions containing arbitrary functions in order to have some degrees of freedom in solving initial and/or boundary value problems of physical interest. This is also achieved through the use of some results known in literature as substitution principles or of the recent Bogoyavlenskij symmetries. Thus, let us give a brief sketch of the involved theorems [10, 11, 13, 19].

In [10], it has been proved that the steady ideal magneto-gas-dynamics equations, with a separable equation of state (which includes (2)), are invariant with respect to the following family of transformations:

$$\mathbf{v}^* = \frac{\mathbf{v}}{m(\mathbf{x})}, \qquad p^* = p, \qquad \mathbf{H}^* = \mathbf{H}, \qquad s^* = [m(\mathbf{x})]^2 s,$$

$m = m(\mathbf{x})$ being an arbitrary scalar function of the space variables subjected to the conditions

$$\mathbf{v} \cdot \nabla m = \mathbf{H} \cdot \nabla m = 0, \tag{6}$$

which mean that the function $m(\mathbf{x})$ must be constant along each individual streamline and along each individual magnetic line.

Power and Rogers [13] extended this result to the class of unsteady flows having the total magnetic pressure steady. They proved that if

$$\{\mathbf{v}(\mathbf{x}, t), p(\mathbf{x}, t), \mathbf{H}(\mathbf{x}, t), s(\mathbf{x}, t)\}$$

is a solution to the unsteady magneto-gas-dynamics equations with the total magnetic pressure $p + \frac{\mu}{2}\mathbf{H} \cdot \mathbf{H}$ steady, then

$$\left\{\begin{matrix} m(\mathbf{x})\mathbf{v}(\mathbf{x}, mt + g(m)), & p(\mathbf{x}, mt + g(m)), \\ \mathbf{H}(\mathbf{x}, mt + g(m)), & [m(\mathbf{x})]^{-2} s(\mathbf{x}, mt + g(m)) \end{matrix}\right\}$$

where $m(\mathbf{x})$ is a steady differentiable scalar function such that the constraints (6) are verified for all t, and $g(m)$ is an arbitrary function of its argument, is also a solution.

Moreover, in [19] it has been proved a mixed substitution principle (linking steady to unsteady solutions), i.e., if

$$\{v_1(x_1, x_2), v_2(x_1, x_2), p(x_1, x_2), h(x_1, x_2), s(x_1, x_2)\}$$

is a solution set to the steady planar magneto-gas-dynamics equations with the magnetic field transverse to the plane of motion, and the adiabatic index is equal to 2, provided $m(x_1, x_2)$ satisfies the condition

$$v_1 \frac{\partial m}{\partial x_1} + v_2 \frac{\partial m}{\partial x_2} = 0,$$

then

$$\left\{\begin{array}{l}\frac{x_1}{t}+\frac{1}{t}m\left(\frac{x_1}{t},\frac{x_2}{t}\right)v_1\left(\frac{x_1}{t},\frac{x_2}{t}\right),\frac{x_2}{t}+\frac{1}{t}m\left(\frac{x_1}{t},\frac{x_2}{t}\right)v_2\left(\frac{x_1}{t},\frac{x_2}{t}\right),\\ \frac{1}{t^4}p\left(\frac{x_1}{t},\frac{x_2}{t}\right),\frac{1}{t^2}h\left(\frac{x_1}{t},\frac{x_2}{t}\right),\left[m\left(\frac{x_1}{t},\frac{x_2}{t}\right)\right]^{-2}s\left(\frac{x_1}{t},\frac{x_2}{t}\right)\end{array}\right\}$$

is a solution set to the unsteady planar equations with the magnetic field transverse to the plane of motion.

The substitution principle for steady incompressible magneto-gas-dynamics [10] is a special subcase of the family of transformations found by Bogoyavlenskij [11] for the magnetohydrodynamics equilibrium equations in the case where the subsonic dynamics of plasma is considered (in such a case the condition $\nabla\cdot\mathbf{v}=0$ is satisfied with high accuracy and this implies, by using the continuity equation, that the plasma density ρ is constant along the streamlines). The main result proved in [11] can be stated as follows.

Let $\rho(\mathbf{x})$, $\mathbf{H}(\mathbf{x})$, $\mathbf{v}(\mathbf{x})$, $p(\mathbf{x})$ be an arbitrary solution of ideal magneto-gas-dynamics for which $\nabla\cdot\mathbf{v}=0$. A new solution is given by

$$\rho_1(\mathbf{x})=a^2(\mathbf{x})\rho(\mathbf{x}),\qquad \mathbf{v}_1(\mathbf{x})=\frac{c(\mathbf{x})}{a(\mathbf{x})}\sqrt{\frac{\mu}{\rho(\mathbf{x})}}\mathbf{H}(\mathbf{x})+\frac{b(\mathbf{x})}{a(\mathbf{x})}\mathbf{v}(\mathbf{x}),$$

$$\mathbf{H}_1(\mathbf{x})=b(\mathbf{x})\mathbf{H}(\mathbf{x})+c(\mathbf{x})\sqrt{\frac{\rho(\mathbf{x})}{\mu}}\mathbf{v}(\mathbf{x}),\qquad p_1(\mathbf{x})=Cp(\mathbf{x})+\frac{\mu}{2}\left(C\mathbf{H}^2(\mathbf{x})-\mathbf{H}_1^2(\mathbf{x})\right)$$

where $a(\mathbf{x}), b(\mathbf{x}), c(\mathbf{x})$ and $\rho(\mathbf{x})>0$ are arbitrary smooth functions that are constant along the magnetic field lines and on the plasma streamlines. Moreover, the functions $b(\mathbf{x})$ and $c(\mathbf{x})$ satisfy the condition

$$b^2(\mathbf{x})-c^2(\mathbf{x})=C=\text{const}.$$

It is easy to recognize that the substitution principle (which holds true also in the case of non-solenoidal velocity fields) is recovered by choosing $a(\mathbf{x})=m(\mathbf{x})$, $b(\mathbf{x})\equiv 1$ and $c(\mathbf{x})\equiv 0$.

3. Steady equations

Let us consider a 2D steady flow. From (5), when $n=2$, we have the invariance of equations (1) with respect to six Lie groups for which the non-zero commutators are

$$[\Xi_1,\Xi_3]=\Xi_1,\qquad [\Xi_2,\Xi_3]=\Xi_2,\qquad [\Xi_1,\Xi_6]=-\Xi_2,\qquad [\Xi_2,\Xi_6]=\Xi_1.$$

To transform the governing system into an equivalent autonomous form [22] we need two commuting infinitesimal operators: thus, we consider two linearly independent combinations of the admitted operators $\Xi_1,\ldots,\Xi_6$:

$$\Xi_A=\sum_{i=1}^{6}\alpha_i\Xi_i,\qquad \Xi_B=\sum_{i=1}^{6}\beta_i\Xi_i, \tag{7}$$

with α_i,β_i $(i=1,\ldots,6)$ constants. If the conditions

$$\alpha_1\beta_3-\alpha_3\beta_1+\alpha_2\beta_6-\alpha_6\beta_2=0,\qquad \alpha_1\beta_6-\alpha_6\beta_1-\alpha_2\beta_3+\alpha_3\beta_2=0 \tag{8}$$

hold true, then the operators Ξ_A and Ξ_B span a two-dimensional Abelian Lie algebra.

As a consequence [21, 22], we are able to introduce the following variable transformation (provided that $\Delta=\alpha_6\beta_3-\alpha_3\beta_6\neq 0$):

$$X_1 = \sum_{i=1}^{2} \frac{a_i\phi_i}{\Delta}, \qquad X_2 = \sum_{i=1}^{2} \frac{b_i\phi_i}{\Delta},$$

$$v_1 = \exp\left(\sum_{i=1}^{2} \frac{c_i\phi_i}{\Delta}\right)(\phi_3 V_1 - \phi_4 V_2), \qquad v_2 = \exp\left(\sum_{i=1}^{2} \frac{c_i\phi_i}{\Delta}\right)(\phi_4 V_1 + \phi_3 V_2),$$

$$H_1 = \exp\left(\sum_{i=1}^{2} \frac{d_i\phi_i}{\Delta}\right)(\phi_3 \widehat{H}_1 - \phi_4 \widehat{H}_2), \qquad H_2 = \exp\left(\sum_{i=1}^{2} \frac{d_i\phi_i}{\Delta}\right)(\phi_4 \widehat{H}_1 + \phi_3 \widehat{H}_2),$$

$$p = \exp\left(2\sum_{i=1}^{2} \frac{d_i\phi_i}{\Delta}\right) P, \qquad s = \exp\left(\sum_{i=1}^{2} \frac{e_i\phi_i}{\Delta}\right) S,$$

where

$$\begin{aligned}
a_1 &= \beta_3, \quad b_1 = \alpha_3, & a_2 &= \beta_6, \quad b_2 = \alpha_6,\\
c_1 &= \alpha_3(\beta_4+\beta_5) - \beta_3(\alpha_4+\alpha_5), & c_2 &= \alpha_6(\beta_4+\beta_5) - \beta_6(\alpha_4+\alpha_5),\\
d_1 &= \frac{\Gamma(\alpha_3\beta_4 - \alpha_4\beta_3)}{\Gamma - 1}, & d_2 &= \frac{\Gamma(\alpha_6\beta_4 - \alpha_4\beta_6)}{\Gamma - 1}\\
e_1 &= 2(\alpha_5\beta_3 - \alpha_3\beta_5), & e_2 &= 2(\alpha_5\beta_6 - \alpha_6\beta_5),\\
k_1 &= \frac{\alpha_2\alpha_6 - \alpha_1\alpha_3}{\alpha_3^2 + \alpha_6^2} & k_2 &= \frac{-\alpha_2\alpha_3 - \alpha_1\alpha_6}{\alpha_3^2 + \alpha_6^2}
\end{aligned}$$

and ϕ_i $(i = 1, \ldots, 4)$ given by

$$\phi_1 = \arctan\left(\frac{x_2 - k_2}{x_1 - k_1}\right), \qquad \phi_2 = \ln r, \qquad \phi_3 = \frac{x_1 - k_1}{r}, \qquad \phi_4 = \frac{x_2 - k_2}{r},$$

with $r = \sqrt{(x_1 - k_1)^2 + (x_2 - k_2)^2}$.

By using X_1, X_2 as the new independent variables, and V_1, V_2, $\widehat{H}_1$, $\widehat{H}_2$, P and S as the new dependent variables, the system (1) writes in a new equivalent autonomous form, i.e.,

$$\frac{\partial(a_{(1}V_{2)}P^{1/\Gamma})}{\partial X_1} + \frac{\partial(b_{(1}V_{2)}P^{1/\Gamma})}{\partial X_2} + \left(c_{(1}V_{2)} + \frac{2}{\Gamma}d_{(1}V_{2)} + \Delta V_1\right)P^{1/\Gamma} = 0,$$

$$\begin{aligned}
&P^{1/\Gamma}S\left(a_{(1}V_{2)}\frac{\partial V_1}{\partial X_1} + b_{(1}V_{2)}\frac{\partial V_1}{\partial X_2} + \left(c_{(1}V_{2)}V_1 - \Delta V_2^2\right)\right) + a_2\frac{\partial P}{\partial X_1} + b_2\frac{\partial P}{\partial X_2}\\
&\qquad + 2d_2P - \mu\widehat{H}_2\left(\frac{\partial(a_1\widehat{H}_1 - a_2\widehat{H}_2)}{\partial X_1} + \frac{\partial(b_1\widehat{H}_1 - b_2\widehat{H}_2)}{\partial X_2}\right.\\
&\qquad \left. + d_1\widehat{H}_1 - d_2\widehat{H}_2 - \Delta\widehat{H}_2\right) = 0,
\end{aligned}$$

$$\begin{aligned}
&P^{1/\Gamma}S\left(a_{(1}V_{2)}\frac{\partial V_2}{\partial X_1} + b_{(1}V_{2)}\frac{\partial V_2}{\partial X_2} + (c_{(1}V_{2)} + \Delta V_1)V_2\right) + a_1\frac{\partial P}{\partial X_1} + b_1\frac{\partial P}{\partial X_2}\\
&\qquad + 2d_1P + \mu\widehat{H}_1\left(\frac{\partial(a_1\widehat{H}_1 - a_2\widehat{H}_2)}{\partial X_1} + \frac{\partial(b_1\widehat{H}_1 - b_2\widehat{H}_2)}{\partial X_2}\right.\\
&\qquad \left. + d_1\widehat{H}_1 - d_2\widehat{H}_2 - \Delta\widehat{H}_2\right) = 0,
\end{aligned}$$

$$\begin{aligned}
&a_1\left(\widehat{H}_1\frac{\partial V_2}{\partial X_1} - \widehat{H}_2\frac{\partial V_1}{\partial X_1}\right) + b_1\left(\widehat{H}_1\frac{\partial V_2}{\partial X_2} - \widehat{H}_2\frac{\partial V_1}{\partial X_2}\right) + a_{(1}V_{2)}\frac{\partial\widehat{H}_1}{\partial X_1}\\
&\qquad + b_{(1}V_{2)}\frac{\partial\widehat{H}_1}{\partial X_2} + \widehat{H}_1(d_{(1}V_{2)} + \Delta V_1) + c_1(\widehat{H}_1V_2 - \widehat{H}_2V_1) = 0,
\end{aligned}$$

$$a_2\left(\widehat{H}_2\frac{\partial V_1}{\partial X_1}-\widehat{H}_1\frac{\partial V_2}{\partial X_1}\right)+b_2\left(\widehat{H}_2\frac{\partial V_1}{\partial X_2}-\widehat{H}_1\frac{\partial V_2}{\partial X_2}\right)+a_{(1}V_{2)}\frac{\partial \widehat{H}_2}{\partial X_1}$$
$$+b_{(1}V_{2)}\frac{\partial \widehat{H}_2}{\partial X_2}+\widehat{H}_2(d_{(1}V_{2)}+\Delta V_2)+c_2(\widehat{H}_2V_1-\widehat{H}_1V_2)=0,$$
$$\frac{\partial(a_{(1}\widehat{H}_{2)})}{\partial X_1}+\frac{\partial(b_{(1}\widehat{H}_{2)})}{\partial X_2}+(d_{(1}\widehat{H}_{2)}+\Delta\widehat{H}_1)=0,\qquad a_{(1}V_{2)}\frac{\partial S}{\partial X_1}+b_{(1}V_{2)}\frac{\partial S}{\partial X_2}+e_{(1}V_{2)}S=0, \tag{9}$$

where we used the notation $w_{(1}z_{2)}=w_1z_2+w_2z_1$.

The system (9) has a more complicated structure than the original one, but we may take advantage of this by looking for simple solutions; coming back to the true physical variables, we finally determine a solution to the original system.

For instance, searching for solutions to the system (9) such that $V_1=0, V_2=v_0$ (v_0 constant), $\widehat{H}_1=0$, we obtain the following solution to the system (1):

$$\begin{aligned} &v_1=-v_0(x_2-k_2)r^{\alpha-1}, && v_2=v_0(x_1-k_1)r^{\alpha-1},\\ &H_1=-(x_2-k_2)r^{-1}\chi(r), && H_2=(x_1-k_1)r^{-1}\chi(r),\\ &p=\Psi(r), && s=\frac{\Psi'+\mu\chi(r^{-1}\chi+\chi')}{v_0^2\Psi^{1/\Gamma}}r^{1-2\alpha}, \end{aligned}\tag{10}$$

where α is an arbitrary constant, while $\chi(r)$ and $\Psi(r)$ are arbitrary functions of r that should be positive and increasing in order to have physically meaningful solutions; moreover the prime $'$ denotes, here and in the following, differentiation with respect to the argument.

A new exact solution is recovered by applying the Smith's substitution principle to the previous solution. In fact, by solving the constraint

$$(x_2-k_2)\frac{\partial m}{\partial x_1}-(x_1-k_1)\frac{\partial m}{\partial x_2}=0,$$

whereupon it follows $m=M(r)$, with $M(r)$ being an arbitrary function of its argument, the following new steady solution is found:

$$\begin{aligned} &v_1=-(x_2-k_2)\Phi(r), && v_2=(x_1-k_1)\Phi(r),\\ &H_1=-(x_2-k_2)r^{-1}\chi(r), && H_2=(x_1-k_1)r^{-1}\chi(r),\\ &p=\Psi(r), && s=\frac{\Psi'+\mu\chi(r^{-1}\chi+\chi')}{r\Phi^2\Psi^{1/\Gamma}}, \end{aligned}\tag{11}$$

where we set $\Phi(r)=r^{\alpha-1}M(r)$. It is worth noting that the use of the Bogoyavlenskij symmetries on the solution (10) gives the same solution (11).

If $\alpha_6=\beta_6=0$, whereupon conditions (8) imply $\alpha_3=\beta_3=0$, we may apply the same procedure (construction of the invertible map and of a new autonomous system). By searching the constant solutions to the transformed system, the following solution to the original system is recovered:

$$\begin{aligned} &v_1=-m_1v_0\exp(\zeta), && v_2=l_1v_0\exp(\zeta), && p=\Psi(\zeta),\\ &H_1=-m_1\chi(\zeta), && H_2=l_1\chi(\zeta), && s=s_0\exp(-2\zeta) \end{aligned}\tag{12}$$

with $v_0, s_0>0, l_1, m_1$ being arbitrary constants, and $\chi(\zeta)$ and $\Psi(\zeta)$ being arbitrary functions of $\zeta=l_1x_1+m_1x_2$ such that $\Psi(\zeta)$ is positive and $\Psi+\frac{\mu}{2}\left(l_1^2+m_1^2\right)\chi^2$ is constant. By applying again the Smith's substitution principle, we recover a new steady solution:

$$\begin{aligned} &v_1=-m_1\Phi(\zeta), && v_2=l_1\Phi(\zeta), && p=\Psi(\zeta),\\ &H_1=-m_1\chi(\zeta), && H_2=l_1\chi(\zeta), && s=\frac{s_0v_0^2}{\Phi^2(\zeta)}, \end{aligned}\tag{13}$$

$\Phi(\zeta)$ being an arbitrary function of ζ. Also in this case, the use of Bogoyavlenskij symmetries on solution (12) gives the same result as that of the substitution principle.

In the 3D steady case it is not possible to construct three independent Lie groups with commuting operators, and so we cannot use the procedure employed above. Nevertheless, we may give 3D extensions of the 2D solutions recovered above. The natural extension of the solution (10), written in compact form, is

$$\begin{aligned} &\mathbf{v} = \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{k}) R^{\alpha-1}, && \mathbf{H} = \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{k}) R^{-1} \chi(R), \\ &p = \Psi(R), && s = \frac{\Psi' + \mu\chi(R^{-1}\chi + \chi')}{|\boldsymbol{\omega} \times (\mathbf{x} - \mathbf{k})|^2 R^{2\alpha-3} \Psi^{1/\Gamma}}, \end{aligned} \tag{14}$$

where $\boldsymbol{\omega} = [-z_0, w_0, v_0]^T$ and $\mathbf{k} = [k_1, k_2, k_3]^T$ are constant vectors, while $\chi(R)$ and $\Psi(R)$ are arbitrary (positive and increasing) functions of

$$\begin{aligned} R = \Big(&\big(v_0^2 + w_0^2\big)(x_1 - k_1)^2 + \big(v_0^2 + z_0^2\big)(x_2 - k_2)^2 + \big(w_0^2 + z_0^2\big)(x_3 - k_3)^2 \\ &+ 2w_0 z_0 (x_1 - k_1)(x_2 - k_2) + 2v_0 (x_3 - k_3)(z_0 (x_1 - k_1) - w_0 (x_2 - k_2))\Big)^{1/2}. \end{aligned}$$

By solving the constraint

$$(\boldsymbol{\omega} \times (\mathbf{x} - \mathbf{k})) \cdot \nabla m = 0,$$

whereupon we find $m = M(R, Q)$, where M is an arbitrary function of R and $Q = \boldsymbol{\omega} \cdot (\mathbf{x} - \mathbf{k})$, we build the new steady solution:

$$\begin{aligned} &\mathbf{v} = \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{k}) \Phi(R, Q), && \mathbf{H} = \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{k}) R^{-1} \chi(R), \\ &p = \Psi(R), && s = \frac{\Psi' + \mu\chi(R^{-1}\chi + \chi')}{R \Phi^2 \Psi^{1/\Gamma}}. \end{aligned} \tag{15}$$

For the solution (14), due to (2), the density is given by

$$\rho = \frac{\Psi' + \mu\chi(R^{-1}\chi + \chi')}{|\boldsymbol{\omega} \times (\mathbf{x} - \mathbf{k})|^2 R^{2\alpha-3}}. \tag{16}$$

Since the solution (14) satisfies the requirements of the Bogoyavlenskij transformations (7), and taking into account that the functions therein involved must be

$$a(\mathbf{x}) = \widehat{a}(R, Q), \qquad b(\mathbf{x}) = \widehat{b}(R, Q), \qquad c(\mathbf{x}) = \widehat{c}(R, Q),$$

with $\widehat{a}$, $\widehat{b}$ and $\widehat{c}$ (such that $\widehat{b}^2(R, Q) - \widehat{c}^2(R, Q) = C = \text{const}$) being arbitrary functions of the indicated arguments, from the solution given by (14) and (16) we immediately generate the new solution

$$\begin{aligned} \rho_1 &= \widehat{a}^2(R, Q)\rho, \\ \mathbf{v}_1 &= \frac{\widehat{c}(R, Q)}{\widehat{a}(R, Q)} \sqrt{\frac{\mu}{\rho}} \mathbf{H} + \frac{\sqrt{C - \widehat{c}^2(R, Q)}}{\widehat{a}(R, Q)} \mathbf{v}, \\ \mathbf{H}_1 &= \sqrt{C + \widehat{c}^2(R, Q)} \mathbf{H} + \widehat{c}(R, Q) \sqrt{\frac{\rho}{\mu}} \mathbf{v}, \\ p_1 &= Cp - \frac{\mu}{2} \left[\widehat{c}^2(R, Q) \left(\mathbf{H}^2 + \frac{\rho}{\mu} \mathbf{v}^2 \right) + 2\widehat{c}(R, Q) \sqrt{C + \widehat{c}^2(R, Q)} \sqrt{\frac{\rho}{\mu}} \mathbf{H} \cdot \mathbf{v} \right], \\ s_1 &= \rho_1 p_1^{-1/\Gamma}, \end{aligned}$$

which is more general than that given in (15).

Also a 3D extension of solution (12) can be built in a similar way.

4. Unsteady equations

Let us consider first the 2D unsteady equations: in order to map the governing equations to an equivalent autonomous form we need three commuting infinitesimal operators; hence, we take three linearly independent combinations of the nine admitted infinitesimal operators:

$$\Xi_A = \sum_{i=1}^{9} \alpha_i \Xi_i, \qquad \Xi_B = \sum_{i=1}^{9} \beta_i \Xi_i, \qquad \Xi_C = \sum_{i=1}^{9} \gamma_i \Xi_i.$$

These operators commute provided that the constants α_i, β_i and γ_i $(i = 1, \ldots, 9)$ satisfy the conditions

$$\begin{array}{lll}
\alpha_{[2}\beta_{4]} - \alpha_{[3}\beta_{9]} + \alpha_{[1}\beta_{7]} = 0, & \alpha_{[2}\beta_{9]} + \alpha_{[3}\beta_{4]} + \alpha_{[1}\beta_{8]} = 0, & \\
\alpha_{[8}\beta_{9]} + \alpha_{[5}\beta_{7]} = 0, & \alpha_{[8}\beta_{5]} + \alpha_{[7}\beta_{9]} = 0, & \alpha_{[1}\beta_{4]} - \alpha_{[1}\beta_{5]} = 0, \\
\alpha_{[2}\gamma_{4]} - \alpha_{[3}\gamma_{9]} + \alpha_{[1}\gamma_{7]} = 0, & \alpha_{[2}\gamma_{9]} + \alpha_{[3}\gamma_{4]} + \alpha_{[1}\gamma_{8]} = 0, & \\
\alpha_{[8}\gamma_{9]} + a_{[5}\gamma_{7]} = 0, & \alpha_{[8}\gamma_{5]} + \alpha_{[7}\gamma_{9]} = 0, & \alpha_{[1}\gamma_{4]} - \alpha_{[1}\gamma_{5]} = 0, \\
\beta_{[2}\gamma_{4]} - \beta_{[3}\gamma_{9]} + \beta_{[1}\gamma_{7]} = 0, & \beta_{[2}\gamma_{9]} + \beta_{[3}\gamma_{4]} + \beta_{[1}\gamma_{8]} = 0, & \\
\beta_{[8}\gamma_{9]} + \beta_{[5}\gamma_{7]} = 0, & \beta_{[8}\gamma_{5]} + \beta_{[7}\gamma_{9]} = 0, & \beta_{[1}\gamma_{4]} - \beta_{[1}\gamma_{5]} = 0,
\end{array}$$

where we have used the notation $w_{[i}z_{j]} = w_i z_j - w_j z_i$ $(i \neq j)$. By using the same procedure as above, we construct an invertible point transformation allowing us to write the system (1), specialized in 2D, to a new equivalent autonomous form.

Non-trivial unsteady solutions can be found when we set $\alpha_4 = \alpha_5 = \alpha_9 = \beta_4 = \beta_5 = \beta_9 = \gamma_4 = \gamma_5 = \gamma_9 = 0$. The variable transformation is

$$\begin{array}{l}
T = \displaystyle\sum_{i=1}^{3} \frac{a_i\phi_i}{\Delta}, \qquad X_1 = \sum_{i=1}^{3} \frac{b_i\phi_i}{\Delta}, \qquad X_2 = \sum_{i=1}^{3} \frac{c_i\phi_i}{\Delta}, \\
v_1 = V_1 + \dfrac{\alpha_7}{\alpha_1} t, \qquad v_2 = V_2 + \dfrac{\alpha_8}{\alpha_1} t, \\
H_1 = \exp\left(\dfrac{\Gamma}{2(\Gamma - 1)} \displaystyle\sum_{i=1}^{3} \frac{d_i\phi_i}{\Delta}\right) \widehat{H}_1, \qquad H_2 = \exp\left(\dfrac{\Gamma}{2(\Gamma - 1)} \displaystyle\sum_{i=1}^{3} \frac{d_i\phi_i}{\Delta}\right) \widehat{H}_2, \\
p = \exp\left(\dfrac{\Gamma}{\Gamma - 1} \displaystyle\sum_{i=1}^{3} \frac{d_i\phi_i}{\Delta}\right) P, \qquad s = \exp\left(\displaystyle\sum_{i=1}^{3} \frac{d_i\phi_i}{\Delta}\right) S,
\end{array}$$

where

$$\begin{array}{llll}
\phi_1 = x_1 - \dfrac{\alpha_7}{2\alpha_1} t^2, & \phi_2 = x_2 - \dfrac{\alpha_8}{2\alpha_1} t^2, & \phi_3 = t, & \\
a_1 = \beta_{[1}\gamma_{3]}, & a_2 = \beta_{[2}\gamma_{1]}, & a_3 = \beta_{[2}\gamma_{3]}, & \\
b_1 = \alpha_{[3}\gamma_{1]}, & b_2 = \alpha_{[1}\gamma_{2]}, & b_3 = \alpha_{[2}\gamma_{3]}, & \\
c_1 = \alpha_{[1}\beta_{3]}, & c_2 = \alpha_{[2}\beta_{1]}, & c_3 = \alpha_{[3}\beta_{2]}, & \Delta = b_2c_1 - b_1c_2, \\
d_i = \alpha_6 a_i + \beta_6 b_i + \gamma_6 c_i, & i = 1, 2, 3, & &
\end{array}$$

and T, X_1, X_2 represent the new independent variables, whereas V_1, V_2, $\widehat{H}_1$, $\widehat{H}_2$, P and S represent the new dependent variables.

A solution is obtained by assuming in the transformed system (whose listing is omitted)

$$\alpha_7 V_1 + \alpha_8 V_2 = c,$$

where c is a constant. In terms of the original variables we have the following solution:

$$
\begin{aligned}
v_1 &= -\alpha_8\Phi(\zeta) + \frac{\alpha_7}{\alpha_1}t, & v_2 &= \frac{c}{\alpha_8} + \alpha_7\Phi(\zeta) + \frac{\alpha_8}{\alpha_1}t,\\
H_1 &= -\alpha_8\chi(\zeta), & H_2 &= \alpha_7\chi(\zeta),\\
p &= \Psi(\zeta), & s &= -\frac{\alpha_1}{\Psi^{1/\Gamma}}\left(\Psi + \frac{\mu}{2}\left(\alpha_7^2+\alpha_8^2\right)\chi^2\right)',
\end{aligned} \tag{17}
$$

where $\Phi(\zeta)$, $\chi(\zeta)$ and $\Psi(\zeta)$ are arbitrary functions of

$$\zeta = \alpha_7x_1 + \alpha_8x_2 - ct - \frac{\alpha_7^2+\alpha_8^2}{2\alpha_1}t^2.$$

In the 3D unsteady case it is not possible to build four linear independent combinations of the admitted operators generating a four-dimensional Abelian Lie algebra. Nevertheless, we may get solutions by extending in a natural way the solution (17); what we get is

$$
\begin{aligned}
\mathbf{v} &= \mathbf{\Phi}\times\boldsymbol{\alpha} + \tfrac{\boldsymbol{\alpha}}{\alpha_1}t + \mathbf{c}, & \mathbf{H} &= \boldsymbol{\chi}\times\boldsymbol{\alpha},\\
p &= \Psi(\zeta), & s &= -\frac{\alpha_1}{\Psi^{1/\Gamma}}\left(\Psi + \frac{\mu}{2}(\boldsymbol{\chi}\times\boldsymbol{\alpha})^2\right)',
\end{aligned} \tag{18}
$$

where $\boldsymbol{\alpha} = [\alpha_7, \alpha_8, \alpha_9]^T$, $\mathbf{c} = [0, c, 0]^T$, $\mathbf{\Phi} = [0, \Phi_3, \Phi_2]^T$, $\boldsymbol{\chi} = [0, \chi_3, \chi_2]^T$ with $\Phi_2(\zeta)$, $\Phi_3(\zeta)$, $\chi_2(\zeta)$, $\chi_3(\zeta)$ and $\Psi(\zeta)$ being arbitrary functions of

$$\zeta = \boldsymbol{\alpha}\cdot\mathbf{x} - ct - \frac{|\boldsymbol{\alpha}|^2}{2\alpha_1}t^2,$$

and c is a constant.

However, it should be remarked that the solutions (17) and (18) exhibit a blow up at least in the velocity field and, moreover, do not satisfy the requirements of the substitution principle for unsteady magneto-gas-dynamics.

5. Plane motion with a transverse magnetic field

A special case occurs when we consider a plane motion of a fluid with the adiabatic exponent Γ is equal to 2 and subjected to a transverse magnetic field. Also in this case we consider three commuting operators by taking three linearly independent combinations of the operators Ξ_i $(i = 1, \ldots, 10)$, where the parameters α_{10}, β_{10} and γ_{10} are non-vanishing.

In the following, in order to simplify the calculations without losing in generality, we take, when it is possible, $\alpha_i = \beta_i = \gamma_i = 0$ $(i = 1, 2, 3, 7, 8)$: this means that we neglect space and time translations and the Galilean transformations. This is not limiting since we may include time and space translations and Galilean transformations directly in the solutions we shall find by simply substituting t with $t - k_0$, x_1 with $x_1 - k_1t - k_2$, x_2 with $x_2 - k_3t - k_4$, v_1 with $v_1 - k_1$ and v_2 with $v_2 - k_2$ where k_i $(i = 0, \ldots, 4)$ are arbitrary constants.

According to the choices of the parameters α_i, β_i and γ_i $(i = 1, \ldots, 9)$ we must distinguish various cases. If the constants α_i, β_i, γ_i $(i = 4, 5, 9)$ are not zero, the following transformation can be introduced:

$$
\begin{aligned}
&T = \sum_{i=1}^{3}\frac{a_i\phi_i}{\Delta}, \qquad X_1 = \sum_{i=1}^{3}\frac{b_i\phi_i}{\Delta}, \qquad X_2 = \sum_{i=1}^{3}\frac{c_i\phi_i}{\Delta},\\
&v_1 = \frac{1}{at^2+bt+c}(x_1V_1 - x_2V_2 + \alpha_{10}x_1t),\\
&v_2 = \frac{1}{at^2+bt+c}(x_2V_1 + x_1V_2 + \alpha_{10}x_2t),
\end{aligned}
$$

$$p = \frac{r^4}{(at^2+bt+c)^4}\left(2\exp\left(\sum_{i=1}^{3}\frac{d_i\phi_i}{\Delta}\right)\right)P,$$

$$h = \frac{r^2}{(at^2+bt+c)^2}\left(\exp\left(\sum_{i=1}^{3}\frac{d_i\phi_i}{\Delta}\right)\right)H, \qquad s = \exp\left(\sum_{i=1}^{3}\frac{d_i\phi_i}{\Delta}\right)S, \tag{19}$$

where

$$a_1 = \beta_{[4}\gamma_{5]}, \qquad a_2 = \beta_{[9}\gamma_{4]} - \beta_{[9}\gamma_{5]}, \qquad a_3 = 2\beta_{[4}\gamma_{9]},$$

$$b_1 = \alpha_{[5}\gamma_{4]}, \qquad b_2 = \alpha_{[4}\gamma_{9]} - \alpha_{[5}\gamma_{9]}, \qquad b_3 = 2\alpha_{[4}\gamma_{5]},$$

$$c_1 = \alpha_{[4}\beta_{5]}, \qquad c_2 = \alpha_{[9}\beta_{4]} - \alpha_{[9}\beta_{5]}, \qquad c_3 = 2\alpha_{[4}\beta_{5]},$$

$$d_i = \alpha_6 a_i + \beta_6 b_i + \gamma_6 c_i, \qquad i = 1, 2, 3,$$

$$\phi_1 = \arctan\frac{x_2}{x_1}, \qquad \phi_2 = \ln\left(\frac{r}{\sqrt{at^2+bt+c}}\right) - \tau, \qquad \phi_3 = \tau,$$

and

$$\tau = \frac{b}{\sqrt{4ac-b^2}}\arctan\left(\frac{2at+b}{\sqrt{4ac-b^2}}\right), \qquad r = \sqrt{x_1^2+x_2^2},$$

with $a = \alpha_{10}, b = \alpha_4 - \alpha_5, c = \alpha_1$, along with the constraints $a > 0, 4ac - b^2 > 0$ and $\Delta = (b_2c_3 - b_3c_2)/\alpha_9 \neq 0$.

By taking T, X_1 and X_2 as the new independent variables, and V_1, V_2, P, H and S as the new dependent variables, we transform the governing system to a new equivalent autonomous form that we omit to write here.

A solution may be obtained by assuming V_1 and H to be constant. The solution, written in terms of the original variables, is

$$v_1 = \frac{(2at+b)x_1 - 2x_2\Phi(\zeta)}{2(at^2+bt+c)}, \qquad v_2 = \frac{2x_1\Phi(\zeta) + (2at+b)x_2}{2(at^2+bt+c)},$$

$$p = \frac{\Psi(\zeta)}{(at^2+bt+c)^2}, \qquad h = \frac{\chi(\zeta)}{at^2+bt+c}, \tag{20}$$

$$s = \frac{8}{4(\Phi^2 - ac) + b^2}\,\frac{\Psi' + \mu\chi\chi'}{\sqrt{\Psi}},$$

where $\Phi(\zeta)$, $\chi(\zeta)$ and $\Psi(\zeta)$ are arbitrary functions of

$$\zeta = \frac{r^2}{at^2+bt+c}.$$

By choosing $\Phi(\zeta) = 0, \Psi(\zeta) = \Psi_0\zeta^{-2}$ and $\chi(\zeta) = \chi_0\zeta^{-1}$ (Ψ_0, χ_0 being arbitrary positive constants), we obtain a solution where the total magnetic pressure is steady; if we solve the constraint

$$x_1\frac{\partial m}{\partial x_1} + x_2\frac{\partial m}{\partial x_2} = 0,$$

whereupon it follows $m = M(x_2/x_1)$ (M arbitrary function of its argument), the use of the substitution principle for unsteady flows [13] allows us to obtain the following new unsteady solution:

$$v_1 = \frac{[2a(Mt+G)+b]x_1 M}{2[a(Mt+G)^2+b(Mt+G)+c]}, \qquad p = \frac{\Psi_0}{r^4},$$
$$v_2 = \frac{[2a(Mt+G)+b]x_2 M}{2[a(Mt+G)^2+b(Mt+G)+c]}, \qquad h = \frac{\chi_0}{r^2}, \tag{21}$$
$$s = \frac{16}{4ac-b^2}\left(\Psi_0 + \frac{\mu}{2}\chi_0^2\right)\frac{[a(Mt+G)^2+b(Mt+G)+c]^2}{\sqrt{\Psi_0} r^4 M^2},$$

where $G(x_2/x_1)$ is also an arbitrary function of its argument. By taking account of space translations and Galilean transformations, we also have the solution

$$v_1 = \frac{[2a(\widehat{M}t+\widehat{G})+b](x_1-k_1t-k_2)}{2[a(\widehat{M}t+\widehat{G})^2+b(\widehat{M}t+\widehat{G})+c]}\widehat{M}+k_1, \qquad p = \frac{\Psi_0}{\widehat{r}^4},$$
$$v_2 = \frac{[2a(\widehat{M}t+\widehat{G})+b](x_2-k_3t-k_4)}{2[a(\widehat{M}t+\widehat{G})^2+b(\widehat{M}t+\widehat{G})+c]}\widehat{M}+k_3, \qquad h = \frac{\chi_0}{\widehat{r}^2},$$
$$s = \frac{16}{4ac-b^2}\left(\Psi_0 + \frac{\mu}{2}\chi_0^2\right)\frac{[a(\widehat{M}t+\widehat{G})^2+b(\widehat{M}t+\widehat{G})+c]^2}{\sqrt{\Psi_0}\widehat{r}^4\widehat{M}^2},$$

where $\widehat{M}$ and $\widehat{G}$ are the arbitrary functions of $(x_2-k_3t-k_4)/(x_1-k_1t-k_2)$, and

$$\widehat{r} = \sqrt{\left(\frac{x_1}{t}-k_1\right)^2 + \left(\frac{x_2}{t}-k_2\right)^2}. \tag{22}$$

Moreover, if we set $a = b = 0$ into solution (20) (that becomes steady), we are able to use the mixed substitution principle [19] and hence generate the new unsteady solution:

$$v_1 = \frac{x_1}{t} - \frac{1}{t}\left(\frac{x_2}{t}-k_2\right)\widehat{M}(\widehat{r}), \qquad v_2 = \frac{x_2}{t} + \frac{1}{t}\left(\frac{x_1}{t}-k_1\right)\widehat{M}(\widehat{r}),$$
$$p = \frac{\widehat{\Psi}(\widehat{r})}{t^4}, \qquad h = \frac{\widehat{\chi}(\widehat{r})}{t^2}, \qquad s = \frac{\widehat{\Psi}' + \mu\widehat{\chi}\widehat{\chi}'}{\widehat{M}^2\sqrt{\widehat{\Psi}}},$$

where $\widehat{M}(\widehat{r})$, $\widehat{\Psi}(\widehat{r})$ and $\widehat{\chi}(\widehat{r})$ are the arbitrary functions of $\widehat{r}$ given by (22). The latter solution has an apparent singularity at $t = 0$; but if we take into account that the governing equations are independent with respect to time translation, then we may substitute t with $t+t_0$ (t_0 arbitrary constant) and remove the singularity.

Now we look for the exact solutions when we assume $\alpha_9 = \alpha_4$ and $\alpha_5 \neq \beta_5 \neq \gamma_5 \neq 0$. In this case, if we want an explicit transformation we need to choose $\alpha_1 = \beta_1 = \gamma_1 = 0$ and $\alpha_i \neq 0, \beta_i \neq 0, \gamma_i \neq 0$ $(i = 2, 3)$. By assuming the transformed velocity non-constant whose components are linked by the relation $c_1V_1 + c_2V_2 = 0$ (c_1 and c_2 constants), we find the solution

$$v_1 = \frac{c_1(\alpha_{10}x_1+\alpha_7) - c_2\Phi(\zeta)}{c_1(\alpha_{10}t-\alpha_5)}, \qquad v_2 = \frac{(\alpha_{10}x_2+\alpha_8)+\Phi(\zeta)}{\alpha_{10}t-\alpha_5},$$
$$p = \frac{p_0}{(\alpha_{10}t-\alpha_5)^4}, \qquad h = \frac{h_0}{(\alpha_{10}t-\alpha_5)^2}, \qquad s = S(\zeta), \tag{23}$$

where p_0 and h_0 are the arbitrary constants, whereas $V(\zeta)$ and $S(\zeta)$ are the arbitrary functions of

$$\zeta = \frac{c_1(\alpha_{10}x_1+\alpha_7) + c_2(\alpha_{10}x_2+\alpha_8)}{\alpha_{10}t-\alpha_5}.$$

The subcase of the previous one that we must analyse to cover all the situations is the case in which $\alpha_5 = \beta_5 = \gamma_5 = 0$. By searching for the solutions of the transformed system such that $\alpha_3 V_1 + \alpha_2 V_2 = c_0$ (c_0 non-vanishing constant), we get

$$
\begin{aligned}
v_1 &= \frac{\Phi(\zeta)}{t} + \frac{(\alpha_{10}x_1 + \alpha_7)t + \alpha_2}{\alpha_{10}t^2},\\
v_2 &= \frac{c_0 - \alpha_2\Phi(\zeta)}{\alpha_3 t} + \frac{(\alpha_{10}x_2 + \alpha_8)t + \alpha_3}{\alpha_{10}t^2},\\
h &= \frac{\chi(\zeta)}{t^2}, \qquad p = \frac{\Psi(\zeta)}{t^4}, \qquad s = \frac{\alpha_{10}(\Psi' + \mu\chi\chi')}{\sqrt{\Psi}},
\end{aligned}
\tag{24}
$$

where $\Phi(\zeta)$, $\chi(\zeta)$ and $\Psi(\zeta)$ (such that p and s are positive) are the arbitrary functions of

$$
\zeta = \frac{\alpha_2 x_1 + \alpha_3 x_2}{t} + \frac{\alpha_2\alpha_7 + \alpha_3\alpha_8}{\alpha_{10}t} + \frac{c_0}{t} + \frac{\alpha_2^2 + \alpha_3^2}{2\alpha_{10}t^2}.
$$

If $\alpha_3 = 0$, one more solution is provided:

$$
\begin{aligned}
v_1 &= -\frac{d_3}{d_2 t} + \frac{(\alpha_{10}x_1 + \alpha_7)t + \alpha_2}{\alpha_{10}t^2}, \qquad v_2 = \frac{v_{2_0}}{t} + \frac{\alpha_{10}x_2 + \alpha_8}{\alpha_{10}t},\\
p &= \frac{p_0}{t^4}\exp(2d_2\zeta), \qquad h = \frac{h_0}{t^2}\exp(d_2\zeta),\\
s &= \frac{-2d_2\alpha_{10}}{\alpha_2\sqrt{p_0}}\left(p_0 + \mu\frac{h_0^2}{2}\right)\exp(d_2\zeta),
\end{aligned}
\tag{25}
$$

where

$$
\zeta = \frac{\alpha_{10}x_1 + \alpha_7}{t} + \frac{\alpha_2}{2\alpha_{10}t^2} - \frac{d_3}{d_2 t},
$$

and v_{2_0}, p_0 and h_0 are the arbitrary constants such that p and s are positive. The same reasoning as before allows us to remove the apparent singularity in $t = 0$.

6. Linear stability of some steady 2D solutions

In this section, we investigate the linear stability of solution (11) and solution (20) (with $a = b = 0$, $c = 1$) under axisymmetric perturbations; a generalization of classical results quoted in [23] is obtained.

The solution (11) in cylindrical coordinates (r, θ, z) reads

$$
\begin{aligned}
&v_r = v_z = 0, \qquad v_\theta = \Phi(r)r, \qquad H_r = H_z = 0, \qquad H_\theta = \chi(r),\\
&p = \Psi(r), \qquad s = \frac{\Psi' + \mu\chi(r^{-1}\chi + \chi')}{r\Phi^2\Psi^{1/\Gamma}}, \qquad \rho = p^{1/\Gamma}s.
\end{aligned}
\tag{26}
$$

By considering a flow between two rigid coaxial cylinders under the influence of a circular magnetic field, we may specify the arbitrary functions by assigning the field at the boundaries $r = R_1$, $r = R_2$, where R_1 and R_2 are the radii of inner and outer cylinders, respectively.

The previous solution contains a well-known solution [23] (section 83)

$$
\begin{aligned}
&v_r = v_z = 0, \qquad v_\theta = \Phi(r)r, \qquad H_r = H_z = 0, \qquad H_\theta = \chi(r),\\
&\rho = \rho_0 = \text{const.}, \qquad \frac{\partial}{\partial r}\left(\left(p + \frac{\mu}{2}\mathbf{H}\cdot\mathbf{H}\right)\right) = (\rho_0\Phi^2 r - \mu r^{-1}\chi^2),
\end{aligned}
\tag{27}
$$

that is recovered from solution (26) when

$$
\Psi(r) = \int (\rho_0\Phi^2 r - \mu r^{-1}\chi^2)\,\mathrm{d}r - \frac{\mu}{2}\chi^2.
$$

In [29], it is proved that solution (27) is linearly stable for axisymmetric perturbations (see [23]) if the Rayleigh's discriminant

$$E(r) = \frac{\rho_0}{r^3}((r^2\Phi)^2)' - \mu r((\chi r^{-1})^2)' > 0, \qquad \forall r \in [R_1, R_2]; \tag{28}$$

therefore, condition (28), ensuring the linear stability, imposes restrictions on the functions $\Phi(r)$ and $\chi(r)$.

Now let us examine the linear stability of solution (26) (where ρ is not constant). Let us consider the axisymmetric perturbations

$$v_r = e^{i(\lambda z+kt)}u_r(r), \qquad v_\theta = e^{i(\lambda z+kt)}u_\theta(r) + r\Phi(r), \qquad v_z = e^{i(\lambda z+kt)}u_z(r),$$
$$H_r = e^{i(\lambda z+kt)}\tilde{H}_r(r), \qquad H_\theta = e^{i(\lambda z+kt)}\tilde{H}_\theta(r) + \chi(r), \qquad H_z = e^{i(\lambda z+kt)}\tilde{H}_z(r),$$
$$\rho = e^{i(\lambda z+kt)}\tilde{\rho}(r) + \frac{1}{r\Phi^2}(\Psi' + \mu\chi(r^{-1}\chi + \chi')),$$

where k is constant and λ is the wave number of the disturbance in the z-direction. By substituting the perturbed solution into the governing equations and solving the linearized system, after some algebra what remains is

$$k^2((\rho(u_r' + r^{-1}u_r))' - \rho\lambda^2 u_r) = -\lambda^2 E(r)u_r, \tag{29}$$

to be solved with the boundary conditions

$$u_r = 0, \quad \text{at} \quad r = R_1 \quad \text{and} \quad r = R_2.$$

By multiplying (29) by ru_r and integrating by parts over the domain, we arrive at the condition

$$k^2 = \frac{\lambda^2\int_{R_1}^{R_2} E(r)u_r^2 r\,dr}{\int_{R_1}^{R_2} r\rho\left((u_r' + r^{-1}u_r)^2 + \lambda^2 u_r^2\right) dr}.$$

In this case, the Rayleigh's discriminant is given by

$$E(r) = \frac{1}{r^3}(\rho(r^2\Phi)^2)' - \mu r((\chi r^{-1})^2)' = \frac{1}{r^3}(r^3(\Psi' + \mu\chi\chi'))' + 4\mu(\chi r^{-1})^2;$$

as a consequence, solution (26) is linearly stable when $E(r) > 0$, whereupon the linear stability requires restrictions on $\Psi(r)$ and $\chi(r)$ not on $\Phi(r)$ and $\chi(r)$ as in the case considered in [23].

Now, we consider solution (20) where we set $a = b = 0$ and $c = 1$, so recovering a steady solution that in cylindrical coordinates reads

$$v_r = 0, \qquad v_\theta = \Phi(r)r,$$
$$p = \Psi(r), \qquad h = \chi(r), \qquad s = \frac{\Psi' + \mu\chi\chi'}{r\Phi^2\sqrt{\Psi}}, \qquad \rho = p^{1/\Gamma}s. \tag{30}$$

Once again, let us consider an incompressible flow between two rigid coaxial cylinders under the influence of an external applied axial magnetic field. By choosing

$$\Psi(r) = \rho_0\int r\Phi^2\,dr - \frac{\mu}{2}h_0^2, \qquad \chi = h_0 = \text{const},$$

our solution reduces to a well-known solution [23] (section 81):

$$v_r = v_z = 0, \qquad v_\theta = r\Phi(r), \qquad H_r = H_\theta = 0, \qquad H_z = h_0,$$
$$\rho = \rho_0 = \text{const}, \qquad p = \rho_0\int r\Phi^2\,dr - \frac{\mu}{2}h_0^2.$$

It is linearly stable [23] when

$$\mu h_0^2 > \frac{\int_{R_1}^{R_2}(4\rho_0(\Phi(r))^2 - E(r))u_r^2 r\,dr}{\int_{R_1}^{R_2} r\left((u_r' + r^{-1}u_r)^2 + \lambda^2 u_r^2\right) dr}, \tag{31}$$

the Rayleigh's discriminant being

$$E(r) = \frac{\rho_0}{r^3}((r^2\Phi)^2)'.$$

Now, let us study the linear stability of solution (30) with $\chi = h_0$, by using the same boundary conditions considered in [23] (perturbation of radial velocity vanishing at the boundaries).

By substituting the axisymmetric perturbed field variables (as above) in the governing equations and solving the linearized system, we get the condition

$$\left(\rho k^2 - \mu h_0^2\lambda^2\right)^2\left((u_r' + r^{-1}u_r)' - \lambda^2 u_r\right) = -\lambda^2\left(\rho k^2 - \mu h_0^2\lambda^2\right)E(r)u_r - 4\mu h_0^2\lambda^4\Phi^2 u_r,$$

from which, by multiplying by ru_r and integrating by parts over the domain, we get the condition

$$\int_{R_1}^{R_2} r\left(\rho k^2 - \mu h_0^2\lambda^2\right)^2\left((u_r' + r^{-1}u_r)^2 + \lambda^2 u_r^2\right)\mathrm{d}r$$
$$= \lambda^2\int_{R_1}^{R_2}\left(\rho k^2 - \mu h_0^2\lambda^2\right)E(r)u_r^2 r\,\mathrm{d}r + 4\mu h_0^2\lambda^4\int_{R_1}^{R_2} r\Phi^2 u_r^2\,\mathrm{d}r. \tag{32}$$

Since the imaginary part of (32) is zero, we may write (32) as

$$I_1 k^4 - I_2 k^2 - I_3 = 0, \tag{33}$$

where

$$I_1 = \int_{R_1}^{R_2}\rho^2\left((u_r' + r^{-1}u_r)^2 + \lambda^2 u_r^2\right)r\,\mathrm{d}r,$$
$$I_2 = \lambda^2\int_{R_1}^{R_2}\rho\left(2\mu h_0^2\left((u_r' + r^{-1}u_r)^2 + \lambda^2 u_r^2\right) + E(r)u_r\right)r\,\mathrm{d}r,$$
$$I_3 = \mu h_0^2\lambda^4\int_{R_1}^{R_2}\left(\Phi^2 u_r^2 - E(r)u_r^2 - \mu h_0^2\left((u_r' + r^{-1}u_r)^2 + \lambda^2 u_r^2\right)\right)r\,\mathrm{d}r,$$

that provide

$$k^2 = \frac{I_2 \pm \sqrt{I_2^2 + 4I_1 I_3}}{2I_1}.$$

Since $I_1 > 0$, we have both values of k^2 real and positive if $I_2^2 + 4I_1 I_3 > 0$ and $I_3 < 0$. The last condition is satisfied if

$$\mu h_0^2 > \frac{\int_{R_1}^{R_2}(4\rho(\Phi(r))^2 - E(r))u_r^2 r\,\mathrm{d}r}{\int_{R_1}^{R_2}\left((u_r' + r^{-1}u_r)^2 + \lambda^2 u_r^2\right)r\,\mathrm{d}r} = \frac{J_1}{J_2}, \tag{34}$$

where the Rayleigh's discriminant is

$$E(r) = \frac{1}{r^3}(\rho(r^2\Phi)^2)' = \frac{1}{r^3}(r^3\Psi')'. \tag{35}$$

The condition $I_2^2 + 4I_1 I_3$ can be rewritten as

$$4\mu^2 h_0^4\left(J_3^2 - I_1 J_2\right) + 4\mu h_0^2(J_3 J_4 + I_1 J_1) + J_4^2 = 0,$$

where

$$J_3 = \int_{R_1}^{R_2}\rho\left((u_r' + r^{-1}u_r)^2 + \lambda^2 u_r^2\right)\mathrm{d}r, \qquad J_4 = \int_{R_1}^{R_2}\rho E(r) r u_r^2\,\mathrm{d}r.$$

By using (34), we get

$$I_2^2 + 4I_1 I_3 > \frac{(2J_1 J_3 + J_4 J_2)^2}{J_2^2} > 0.$$

7. Concluding remarks

In this paper, the equations of ideal magneto-gas-dynamics have been considered in order to obtain some explicit classes of solutions. The Lie point symmetries admitted by the system at hand were used to conveniently transform the source system into an equivalent autonomous target system. It has been shown how to build non-trivial exact solutions by the direct inspection of the existence of some simple solutions of the transformed system.

The procedure here proposed involves lengthy but straightforward calculations (that can be done easily through the use of a computer algebra system), but may provide a systematic tool for finding classes of exact solutions to nonlinear PDEs.

Use has been made also of some finite transformations like the substitution principles and the Bogoyavlenskij symmetries, whereupon we were able to generate exact solutions involving some arbitrary functions, useful to study physically meaningful initial and/or boundary value problems.

The linear stability with respect to axisymmetric perturbations of two planar steady solutions, generalizing some known solutions considered in [23] for the flow between two rigid coaxial cylinders under the influence of a circular magnetic field or of an axial magnetic field, respectively, has also been investigated.

Various classes of time-dependent solutions have been obtained for planar magneto-gas-dynamics equations with $\Gamma = 2$ and the magnetic field transverse to the plane of motion; future work will be devoted to investigate more carefully these solutions and their stability (see also [30]), as well as to build some more complicated solutions and use them for modelling physically motivated situations.

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